REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. REPORT DATE *(DD-MM-YYYY)*
28-09-2004

2. REPORT TYPE
REPRINT

3. DATES COVERED *(From - To)*

4. TITLE AND SUBTITLE
Using quasi-linear diffusion to model acceleration and loss from wave-particle interactions

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER
621010F

5d. PROJECT NUMBER
1010

5e. TASK NUMBER
RS

5f. WORK UNIT NUMBER
A1

6. AUTHOR(S)
Jay M. Albert

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Air Force Research Laboratory/VSBXR
29 Randolph Road
Hanscom AFB, MA 01731-3010

8. PERFORMING ORGANIZATION REPORT NUMBER
AFRL-VS-HA-TR-2004-1207

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

10. SPONSOR/MONITOR'S ACRONYM(S)

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited.

13. SUPPLEMENTARY NOTES
REPRINTED FROM: Space Weather, Vol. 2, S09S03, doi:10.1029/2004SW000069, 2004

14. ABSTRACT
Current research has reemphasized the importance of cyclotron resonant wave particle interactions for radiation belt electrons. Whistler mode hiss, chaorus, and EMIC waves can act in combination to cause acceleration and loss of radiation belt electrons at greater rates than previously appreciated. These processes can be described by quasi-linear theory, but calculating quasi-linear diffusion coefficients is computationally demanding. Recent advances have een made in computing bounce averaged quasi-linear pitch angle, energy, and mixed diffusion coefficients for hiss and EMIC in the high density plasmasphere; this paper outlines generalization of these techniques for chorus waves, prevalent in the low density region outside the plasmasphere. These coefficients are associated with a two-dimensional diffusion equation whose numerical solution by finite differencing methods requires care, for reasons having to do with the relation between the mixed and other diffusion coefficients, as discussed.

15. SUBJECT TERMS
Energetic particles, precipitating
Energetic particles, trapped
Wave/particle interactions
Magnetosphere--inner
Pitch angle diffusion
Cyclotron resonance

16. SECURITY CLASSIFICATION OF: a. REPORT	b. ABSTRACT	c. THIS PAGE	17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
UNCL	UNCL	UNCL	UNL		Jay M. Albert
					19b. TELEPHONE NUMBER *(Include area code)* (781) 377-3992

AFRL-VS-HA-TR-2004-1207

SPACE WEATHER, VOL. 2, S09S03, doi:10.1029/2004SW000069, 2004

Using quasi-linear diffusion to model acceleration and loss from wave-particle interactions

J. M. Albert

Air Force Research Laboratory, Space Vehicles Directorate, Hanscom Air Force Base, Massachusetts, USA

Received 30 January 2004; revised 1 April 2004; accepted 6 April 2004; published 28 September 2004.

[1] Current research has reemphasized the importance of cyclotron resonant wave particle interactions for radiation belt electrons. Whistler mode hiss, chorus, and EMIC waves can act in combination to cause acceleration and loss of radiation belt electrons at greater rates than previously appreciated. These processes can be described by quasi-linear theory, but calculating quasi-linear diffusion coefficients is computationally demanding. Recent advances have been made in computing bounce averaged quasi-linear pitch angle, energy, and mixed diffusion coefficients for hiss and EMIC in the high density plasmasphere; this paper outlines generalization of these techniques for chorus waves, prevalent in the low density region outside the plasmasphere. These coefficients are associated with a two-dimensional diffusion equation whose numerical solution by finite differencing methods requires care, for reasons having to do with the relation between the mixed and other diffusion coefficients, as discussed. *INDEX TERMS:* 2716 Magnetospheric Physics: Energetic particles, precipitating; 2720 Magnetospheric Physics: Energetic particles, trapped; 7867 Space Plasma Physics: Wave/particle interactions; 2730 Magnetospheric Physics: Magnetosphere—inner; *KEYWORDS:* wave-particle interactions, pitch angle diffusion, diffusion equation, cyclotron resonance

Citation: Albert, J. M. (2004), Using quasi-linear diffusion to model acceleration and loss from wave-particle interactions, *Space Weather*, 2, S09S03, doi:10.1029/2004SW000069.

1. Introduction

[2] Energetic particles in the inner magnetosphere tend to be very stable, because of the three adiabatic invariants of their motion [*Schulz and Lanzerotti*, 1974]. Waves that are resonant with one of the three corresponding frequencies (gyro, bounce, drift) thus provide an important means of affecting the distribution. Quasi-linear theory provides a framework for evaluating the effect of waves on test particles, assuming that the waves have a broad, continuous distribution in frequency and wavenormal angle. It provides diffusion coefficients for pitch angle and energy, which apply to the instantaneous position of the particle and which may be bounce-averaged.

[3] *Lyons et al.* [1972, 1973] applied this approach to plasmaspheric hiss acting on radiation belt electrons. They showed that the pitch angle diffusion coefficients can account for the observed pitch angle distributions. They also showed that precipitation lifetimes derived from the bounce-averaged pitch angle diffusion rates, combined with radial diffusion, reproduces the inner zone/slot/outer zone radial structure. These lifetime calculations were revisited by *Albert* [1994] and compared to decay rates observed by CRRES [*Albert*, 2000a]. Recently the calculations have been redone using more detailed estimates of the wave populations originating in hiss, lightning, and ground-based VLF transmitters [*Abel and Thorne*, 1998; *Albert*, 1999]. Diffusion by hiss has also been included in studies of superthermal electrons [*Liemohn et al.*, 1997], electron beams [*Khazanov et al.*, 1999, 2000], and ring current electrons [*Fok et al.*, 2001; *Khazanov et al.*, 2003a] and protons [*Kozyra et al.*, 1994, 1995]. The Salammbô code [*Beutier and Boscher*, 1995; *Bourdarie et al.*, 1996] combines pitch angle diffusion from hiss with radial diffusion and other sources and losses in a global electron model.

[4] Quasi-linear diffusion has also been used to model the effects of EMIC (electromagnetic ion cyclotron) waves on ring current ions [*Jordanova et al.*, 1996, 1997, 1998, 2001; *Khazanov et al.*, 2002, 2003b] and radiation belt electrons [*Summers et al.*, 1998; *Summers and Thorne*, 2003; *Albert*, 2003]. All of the above calculations treated the waves (hiss or EMIC) in the high density approximation appropriate inside the plasmasphere.

[5] A third important class of waves is known as chorus, which propagates in the whistler mode outside the plasmasphere. Unlike hiss and EMIC, which mostly affect pitch angle, these waves can cause significant diffusion in energy [*Summers et al.*, 1998, 2002; *Summers and Ma*, 2000], and may account for the observed rapid regeneration of MeV electrons in the outer zone following their depletion by magnetic storms [*Meredith et al.*, 2001, 2002, 2003]. Energization is expected to be stronger for lower ratios of the plasma frequency to gyrofrequency, which is large within the plasmasphere but decreases to nearly one outside it. Therefore it is important that the most recent calculations, which confirm this expectation [*Horne et al.*, 2003], are valid for general values of ω_{pe}/Ω_e, even though results using the high density approximation turn out to give surprisingly good agreement (R. Horne, personal

communication, 2003). Nevertheless, it is desirable to be able to evaluate diffusion rates for low density chorus properly. An efficient method for doing this is discussed in section 3, on the basis of extending techniques developed for high density hiss. These techniques avoid spending computation time on wavenormal angles and harmonic numbers for which the resonant frequency does not exist or does not lie between the cutoff frequencies. The next logical step is to use the rates in the diffusion equation for the particle distribution, but including pitch angle, energy, and cross diffusion simultaneously presents additional numerical difficulties. Likely reasons for this, having to do with the general relation between these diffusion coefficients, are discussed in section 4.

[6] It should be mentioned that quasi-linear theory is not the only way to treat cyclotron-resonant wave-particle interactions. For narrowband waves, with strongly spatially varying parameters, it is more appropriate to consider the detailed motion of a particle as it passes through resonance. This approach also has a long history [*Roberts and Buchsbaum*, 1964; *Bell*, 1984], and a Hamiltonian treatment has been given by, e.g., *Albert* [1993, 2000b, 2002]. The relation between the two approaches is discussed by *Swanson* [1989] and by *Albert* [2001].

2. Quasi-linear Diffusion

[7] The cyclotron resonance condition for a diffusing electron is

$$\omega - k_\parallel v_\parallel = -n\Omega_e/\gamma, \tag{1}$$

where $\Omega_e = |e|B/mc$ is the electron's nonrelativistic gyrofrequency, γ is its relativistic factor, and ω is the frequency of the resonant wave. The local pitch angle of the particle is α, and the wavenormal angle, between **k** and **B**, is θ. The parallel wave number $k_\parallel$ is found from the cold plasma index of refraction, $\eta = kc/\omega$, which is determined by ω and θ:

$$\frac{1}{\eta^2} = \left[(RL - PS)\sin^2\theta + 2PS \pm \sqrt{(RL-PS)^2\sin^4\theta + 4P^2D^2\cos^2\theta}\right]/2PRL. \tag{2}$$

The standard wave parameters R, L, P, S, and D are functions of ω [e.g., *Stix*, 1962] but do not depend on θ. The sign $\pm$ is chosen to give the wave mode and polarization of interest.

[8] The local quasi-linear diffusion coefficients are sums of integrals over wavenormal angle θ [*Lyons*, 1974b]:

$$D_{\alpha\alpha} = \sum_n D^n_{\alpha\alpha},\ D_{p\alpha} = D_{\alpha p} = \sum_n D^n_{\alpha p},\ D_{pp} = \sum_n D^n_{pp}, \tag{3}$$

with

$$D^n_{\alpha\alpha} = \int d\theta D^{n\theta}_{\alpha\alpha},\ D^n_{\alpha p} = \int d\theta D^{n\theta}_{\alpha p}, D^n_{pp} = \int d\theta D^{n\theta}_{pp}. \tag{4}$$

The integrands are related by

$$D^{n\theta}_{\alpha p} = \Lambda_n D^{n\theta}_{\alpha\alpha},\ D^{n\theta}_{pp} = \Lambda_n^2 D^{n\theta}_{\alpha\alpha}, \tag{5}$$

where $\Lambda_n = [-\sin\alpha\cos\alpha/(\sin^2\alpha + n\Omega_e/\omega\gamma)]$ [*Lyons*, 1974a]. In principle the sums are over an infinite number of n values, so a significant amount of computation is required, which must be repeated for each value of particle location, energy, and pitch angle. Bounce averaging adds another layer of integration.

[9] Since $D^{n\theta}_{\alpha\alpha}$ depends on both θ and ω, the resonant frequency $\omega(\theta)$ must be found from the resonance condition in order to carry out the θ integrations. $D^{n\theta}_{\alpha\alpha}$ also contains factors to describe the wave magnetic field distributions, which are usually modeled as gaussian in ω and $\tan\theta$ with sharp cutoffs:

$$B^2_{\text{wave}} = \begin{cases} B^2(\omega)g_\omega(\theta), & \theta_1 \le \theta \le \theta_2,\ \omega_{LC} \le \omega \le \omega_{UC} \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

In general, nothing guarantees in advance that the frequency $\omega(\theta)$, once found, will lie between the frequency cutoffs. This can be a drastic source of computational inefficiency, which can be avoided as described below.

[10] The resonance condition can be expressed as

$$\frac{1}{\eta^2} = \frac{v_\parallel^2}{c^2}\frac{\omega^2}{(\omega + n\Omega_e/\gamma)^2}\cos^2\theta, \tag{7}$$

or $V(\omega, \theta) = \Psi(\omega, \theta)$, where from now on V will denote the expression on the right-hand side of equation (7) and Ψ will be used for $1/\eta^2$. For fixed θ, the curve of V versus ω has fairly simple behavior, as illustrated in Figure 1. With a high density approximation to Ψ appropriate within the plasmasphere [*Lyons*, 1974b], the dependence of Ψ on ω is tractable as well, as discussed by *Albert* [1999] for hiss and by *Albert* [2003] for EMIC waves. Figure 2 indicates the behavior of $\Psi(\omega)$ at fixed θ for several values of the density parameter ω_{pe}^2/Ω_e^2.

3. Evaluating the Diffusion Coefficients

[11] It is a simple but powerful observation that, at fixed θ, the curves V and Ψ cannot intersect if the smallest V value is greater than the largest Ψ value, or if the largest V value is less than the smallest Ψ value. That is, there can only be a resonant frequency if

$$V_{\min} < \Psi_{\max} \quad \text{and} \quad V_{\max} > \Psi_{\min}, \tag{8}$$

where the minimum and maximum values are taken over the frequency range ω_{LC} to ω_{UC}. The geometric behavior of the curves allows one to determine these minimum and maximum values. For example, for the case $n > 0$, since $V(\omega)$ is always increasing, $V_{\min}$ is $V(\omega_{LC})$ and $V_{\max}$ is $V(\omega_{UC})$. Similarly, $\Psi_{\min}$ and $\Psi_{\max}$ can be found explicitly, using Ψ appropriate to high density hiss and EMIC waves. It then turns out that conditions

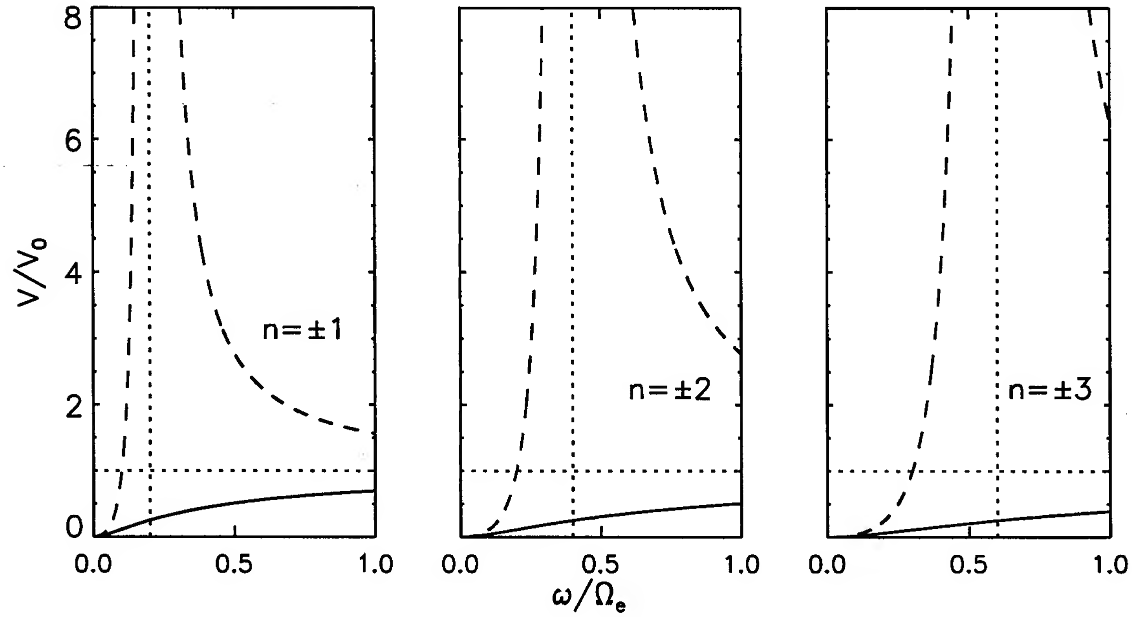

Figure 1. The behavior of the function $V(\omega)$, indicated by plotting V/V_0 against ω/Ω_e, where V_0 is $(v_\parallel^2/c^2)\cos^2\theta$ (see equation 7). The solid lines are for positive values of n and the dashed lines are for negative values of n. Curves with negative n have a singularity at $\omega/\Omega_e = |n|/\gamma$. For large ω, the curves all approach the $n = 0$ value $V = V_0$ (dotted line). Here, for illustration, the relativistic factor γ is 5, corresponding to about a 2 MeV electron.

like those in (8) can be inverted algebraically to give inequalities for $\cos^2\theta$ of the form

$$A\cos^4\theta + B\cos^2\theta + C > 0, \tag{9}$$

where A, B, and C are complicated but fully specified functions of ω_{LC} and ω_{UC}. The corresponding θ values are easily found by considering the zeroes of the quadratic and the sign of A. Only these values will be compatible with the conditions in equation (8); all other θ ranges can be omitted from the integrals for the diffusion coefficients.

[12] Since V decreases as $|n|$ increases, and Ψ is independent of n, there will be a value of $|n|$ large enough that $V_{\max}$ is less than $\Psi_{\min}$ for all θ. This and all larger values of $|n|$ will not contribute any resonances to D at all. This gives a systematic criterion for cutting off the infinite sum over n in equation (3).

[13] The success of this approach hinges on being able to characterize the behavior of $\Psi(\omega)$, so that $\Psi_{\max}$ and $\Psi_{\min}$ can be specified for any general value of θ. As mentioned above, this has previously been carried out for plasmaspheric hiss and EMIC waves. For whistler mode chorus with general ω_{pe}/Ω_e, $\Psi(\omega)$ has the same qualitative shape as in the high density case; this has been verified through extensive numerical evaluation. Therefore $\Psi_{\min}$ is the smaller of $\Psi(\omega_{LC})$ and $\Psi(\omega_{UC})$, but $\Psi_{\max} = \Psi(\omega_{peak})$, which can be found exactly in the high density limit, can only be estimated. One such estimate is the high density value, because $\Psi(\omega) < \Psi^{HD}(\omega)$ so that

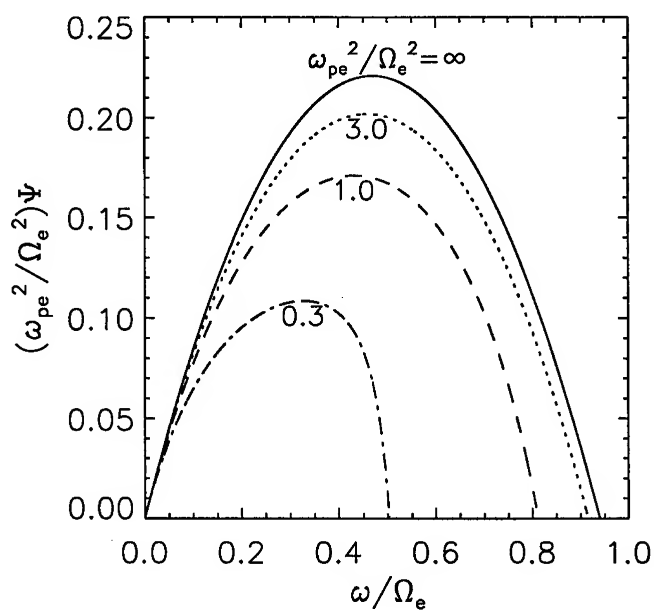

Figure 2. The behavior of the function $\Psi(\omega) = 1/\eta^2$ for fixed θ, indicated by plotting $(\omega_{pe}^2/\Omega_e^2)\Psi$ against ω/Ω_e at $\theta = 20°$ for several values of ω_{pe}^2/Ω_e^2 (see equation 2). The top curve shows the high density approximation appropriate to whistler mode waves within the plasmasphere. The cyclotron resonance condition can be expressed as $V(\omega, \theta) = \Psi(\omega, \theta)$.

$\Psi_{max} < \Psi^{HD}(\omega_{peak})$. More refined upper bounds can be developed, and will be reported in detail elsewhere. With these estimates of Ψ_{min} and Ψ_{max}, the θ ranges in (4) may be reduced, and the diffusion coefficients calculated. This technique does not change the results at all, but allows them to be found much more efficiently, since computation time is not spent trying to obtain resonant frequencies which do not exist or which do not lie between the cutoff values, nor spent considering unnecessarily large values of $\pm n$.

4. Diffusion Coefficients and Diffusion Equations

[14] If $n \neq 0$ and $\omega\gamma \ll \Omega_e$, the definition of Λ_n and equation (5) show that $D_{\alpha\alpha}^{n\theta} \gg |D_{\alpha p}^{n\theta}| \gg D_{pp}^{n\theta}$. Equation (5) also implies

$$D_{\alpha\alpha}^{n\theta} D_{pp}^{n\theta} = \left(D_{\alpha p}^{n\theta}\right)^2. \tag{10}$$

On the other hand, ignoring variations in the coefficients, the condition

$$D_{\alpha\alpha} D_{pp} > D_{\alpha p}^2 \tag{11}$$

must be satisfied in order for the 2D diffusion equation [*Lyons*, 1974b]

$$\begin{aligned}\frac{\partial f}{\partial t} = \frac{1}{p \sin\alpha}\frac{\partial}{\partial\alpha}\sin\alpha\left(D_{\alpha\alpha}\frac{1}{p}\frac{\partial f}{\partial\alpha} + D_{\alpha p}\frac{\partial f}{\partial p}\right) \\ + \frac{1}{p^2}\frac{\partial}{\partial p}p^2\left(D_{p\alpha}\frac{1}{p}\frac{\partial f}{\partial\alpha} + D_{pp}\frac{\partial f}{\partial p}\right)\end{aligned} \tag{12}$$

to be numerically well-behaved [*Richtmyer and Morton*, 1967, *Gourlay and McKee*, 1977].

[15] In general, the integral version of the Cauchy-Schwarz inequality is

$$\left[\int f^2(\theta)d\theta\right]\left[\int g^2(\theta)d\theta\right] \geq \left[\int f(\theta)g(\theta)d\theta\right]^2 \tag{13}$$

for any functions f and g, with equality holding only if f is a constant times g [*Gradshteyn and Ryzhik*, 1980]. Taking $f(\theta) = (D_{\alpha\alpha}^{n\theta})^{1/2}$ and $g(\theta) = (D_{\alpha\alpha}^{n\theta})^{1/2}\Lambda_n$ gives

$$\begin{cases} D_{\alpha\alpha}^{n} D_{pp}^{n} > \left(D_{\alpha p}^{n}\right)^2, & n \neq 0, \\ D_{\alpha\alpha}^{n} D_{pp}^{n} = \left(D_{\alpha p}^{n}\right)^2, & n = 0, \end{cases} \tag{14}$$

since Λ_n depends on θ through ω except when $n = 0$. Summing over n, (11) is satisfied as long as resonances with $n \neq 0$ are present. Otherwise, the diffusion is not truly two dimensional, and equation (12) can be replaced by a 1D diffusion equation in $p_\parallel$ [*Lyons et al.*, 1972].

[16] The bounce-averaged version of the diffusion equation is [e.g., *Kozyra et al.*, 1994]

$$\begin{aligned}\frac{\partial f}{\partial t} = \frac{1}{Gp}\frac{\partial}{\partial\alpha_0}G\left(\hat{D}_{\alpha_0\alpha_0}\frac{1}{p}\frac{\partial f}{\partial\alpha_0} + \hat{D}_{\alpha_0 p}\frac{\partial f}{\partial p}\right) \\ + \frac{1}{G}\frac{\partial}{\partial p}G\left(\hat{D}_{p\alpha_0}\frac{1}{p}\frac{\partial f}{\partial\alpha_0} + \hat{D}_{pp}\frac{\partial f}{\partial p}\right),\end{aligned} \tag{15}$$

where $G = p^2 T(\alpha_0)\sin\alpha_0\cos\alpha_0$ [*Schulz*, 1991] and $T(\alpha_0) \approx 1.30 - 0.56\sin\alpha_0$ is the normalized bounce period [e.g., *Lyons et al.*, 1972]. The bounce averaged diffusion coefficients are

$$\hat{D}_{\alpha_0\alpha_0} = \int D_{\alpha\alpha}\left(\frac{\partial\alpha_0}{\partial\alpha}\right)^2\frac{dt}{\tau_b} = \int D_{\alpha\alpha}W(\lambda)d\lambda, \tag{16}$$

etc., where $W(\lambda) = \cos\alpha\cos^7\lambda / T(\alpha_0)\cos^2\alpha_0$ is the bounce average weighting factor [*Lyons et al.*, 1972]. The bounce-average equivalent of (11) is

$$\hat{D}_{\alpha_0\alpha_0}\hat{D}_{pp} > \left(\hat{D}_{\alpha_0 p}\right)^2. \tag{17}$$

For a single n, a version of the Cauchy-Schwarz inequality can be applied to integrals over λ, with $f(\lambda) = (D_{\alpha\alpha}W)^{1/2}$ and $g(\lambda) = (D_{pp}W)^{1/2}$. Since W depends on λ, (17) is confirmed even for $n = 0$. However, $n = 0$ resonances typically occur near the mirror point, where $v_\parallel / c = 1/\eta_\parallel$ is small, not over a wide range of latitude. Thus if only $n = 0$ resonances are present, the bounce-averaged diffusion coefficients are almost the same as their local, mirror point values and (17) is just barely satisfied, so again the diffusion is nearly one-dimensional (in the second adiabatic invariant J).

[17] Transforming variables from (α_0, p) to some other set (u, v), such as the adiabatic invariants (μ, J), does not change things since

$$D_{uu}D_{vv} - D_{uv}^2 = \left|\frac{\partial(u,v)}{\partial(\alpha_0,p)}\right|^2\left(\hat{D}_{\alpha_0\alpha_0}\hat{D}_{pp} - \hat{D}_{\alpha_0 p}^2\right), \tag{18}$$

so that for nonsingular transformations the 2D condition is the same in either set of variables. As a consequence pure pitch angle diffusion, which violates (17), is problematic when treated as two-dimensional in (μ, J). The Salammbô code, which took this approach, only ran stably when strong radial diffusion was also included (S. Bourdarie, personal communication, 2002) though it has recently been rewritten in (α_0, p) [*Boscher*, 2003]. *Kozyra et al.* [1995] were led to rewrite equation (15) (for nonrelativistic ions) in conservative form and use a finite volume scheme, but the results are inconclusive because of coding errors in evaluating the diffusion coefficients [*Liemohn et al.*, 1997]. Other models have typically treated diffusion as 1D in α_0 [*Jordanova et al.*, 1997; *Khazanov et al.*, 2003b], though some simulations included energy diffusion without the cross term [*Jordanova et al.*, 1998, 2001]. One current effort recasts the 2D diffusion process in (μ, J) as a nonlinear advection

problem and relies on a sophisticated flux-limiting algorithm (M.-C. Fok, personal communication, 2004), but no results have yet been published. Thus the reliable numerical solution of equation (15) or its equivalent remains an open problem.

[18] Treating the diffusion coefficients as constant or slowly varying, stability analysis of standard finite differencing methods [*Richtmyer and Morton*, 1967] shows that the fully explicit scheme should be stable for timestep Δt small enough that

$$\left[\frac{D_{uu}}{(\Delta u)^2}+\frac{D_{vv}}{(\Delta v)^2}\right]\Delta t<\frac{1}{2}, \tag{19}$$

and that the fully implicit and Crank-Nicholson schemes should be unconditionally stable for any choice of variables (u, v). Nevertheless one expects, and experience suggests, that numerical behavior is better with variables that make the "cross term" D_{uv} and its gradients small compared to the larger of the "diagonal terms" D_{uu} and D_{vv}. From this point of view, (α_0, p) is probably preferable to (μ, J) [*Karney*, 1986; *Réveillé et al.*, 2001].

5. Summary

[19] As a description of wave-particle interactions, quasi-linear diffusion is of renewed importance in understanding radiation belt dynamics. The techniques described here make it feasible to compute the diffusion coefficients for low density chorus waves, as well as whistler mode hiss and broadband EMIC waves, by identifying wavenormal angle ranges of waves within a prescribed frequency band that are or are not resonant with specified particles. These coefficients are for diffusion in equatorial pitch angle α_0, momentum p, and mixed diffusion ($D_{\alpha_0 p}$) and are to be used to advance the time-dependent diffusion equation, but this is a surprisingly challenging problem in its own right. It was argued above that a source of numerical problems is that, because of the relation (5) between the different diffusion coefficients, the underlying character of the diffusion can actually be one-dimensional or nearly so; also, working in the variables (μ, J) rather than (α_0, p) is likely to make the numerical stability problems worse. This is unfortunate since (μ, J) is certainly more convenient for including radial diffusion at constant μ and J.

[20] **Acknowledgments.** This work was supported by the Space Vehicles Directorate of the Air Force Research Laboratory and by the Boston College Institute for Scientific Research under USAF contract F19628-00-C-0073. The author thanks S. L. Young for helpful discussions and a careful reading of the manuscript.

References

Abel, B., and R. M. Thorne (1998), Electron scattering loss in Earth's inner magnetosphere: 1. Dominant physical processes, *J. Geophys. Res.*, *103*, 2385.
Albert, J. M. (1993), Cyclotron resonance in an inhomogeneous magnetic field, *Phys. Fluids B*, *5*, 2744.
Albert, J. M. (1994), Quasi-linear pitch-angle diffusion coefficients: Retaining high harmonics, *J. Geophys. Res.*, *99*, 23,741.
Albert, J. M. (1999), Analysis of quasi-linear diffusion coefficients, *J. Geophys. Res.*, *104*, 2429.
Albert, J. M. (2000a), Pitch angle diffusion as seen by CRRES, *Adv. Space Res.*, *25*, 2343.
Albert, J. M. (2000b), Gyroresonant interactions of radiation belt particles with a monochromatic electromagnetic wave, *J. Geophys. Res.*, *105*, 21,191.
Albert, J. M. (2001), Comparison of pitch angle diffusion by turbulent and monochromatic whistler waves, *J. Geophys. Res.*, *106*, 8477.
Albert, J. M. (2002), Nonlinear interaction of outer zone electrons with VLF waves, *Geophys. Res. Lett.*, *29*(8), 1275, doi:10.1029/2001GL013941.
Albert, J. M. (2003), Evaluation of quasi-linear diffusion coefficients for EMIC waves in a multi-species plasma, *J. Geophys. Res.*, *108*(A6), 1249, doi:10.1029/2002JA009792.
Bell, T. F. (1984), The nonlinear gyroresonance interaction between energetic electrons and coherent VLF waves propagating at an arbitrary angle with respect to the Earth's magnetic field, *J. Geophys. Res.*, *89*, 905.
Beutier, T., and D. Boscher (1995), A three-dimensional analysis of the electron radiation belt by the Salammbô code, *J. Geophys. Res.*, *100*, 14,853.
Boscher, D. (2003), Radiation belts modeling using a diffusion equation, paper presented at the International Space Environment Conference, French Natl. Aerospace Res. Estab., Toulouse, France, 2–5 September.
Bourdarie, S., D. Boscher, T. Beutier, J.-A. Sauvaud, and M. Blanc (1996), Magnetic storm modeling in the Earth's electron belt by the Salammbô code, *J. Geophys. Res.*, *101*, 27,171.
Fok, M.-C., T. E. Moore, and W. N. Spjeldvik (2001), Rapid enhancement of radiation belt fluxes due to substorm dipolarization of the geomagnetic field, *J. Geophys. Res.*, *106*, 3873.
Gourlay, A. R., and S. McKee (1977), The construction of hopscotch methods for parabolic and elliptic equations in two space dimensions with a mixed derivative, *J. Comp. Appl. Math.*, *3*, 201.
Gradshteyn, I. S., and I. M. Ryzhik (1980), *Table of Integrals, Series, and Products*, Academic, San Diego, Calif.
Horne, R. B., S. A. Glauert, and R. M. Thorne (2003), Resonant diffusion of radiation belt electrons by whistler-mode chorus, *Geophys. Res. Lett.*, *30*(9), 1493, doi:10.1029/2003GL016963.
Jordanova, V. K., J. U. Kozyra, and A. F. Nagy (1996), Effects of heavy ions on the quasi-linear diffusion coefficients from resonant interactions with electromagnetic ion cyclotron waves, *J. Geophys. Res.*, *101*, 19,771.
Jordanova, V. K., J. U. Kozyra, A. F. Nagy, and G. V. Khazanov (1997), Kinetic model of the ring current-atmosphere interactions, *J. Geophys. Res.*, *102*, 14,279.
Jordanova, V. K., C. J. Farrugia, J. M. Quinn, R. M. Thorne, K. W. Ogilvie, R. P. Lepping, G. Lu, A. J. Lazurus, M. F. Thomsen, and R. D. Belian (1998), Effect of wave-particle interactions on ring current evolution for January 10–11, 1997: Initial results, *Geophys. Res. Lett.*, *25*, 2971.
Jordanova, V. K., C. J. Farrugia, R. M. Thorne, G. V. Khazanov, G. D. Reeves, and M. F. Thomsen (2001), Modeling ring current proton precipitation by electromagnetic ion cyclotron waves during the May 14–16, 1997, storm, *J. Geophys. Res.*, *106*, 7.
Karney, C. F. F. (1986), Fokker-Planck and quasi-linear codes, *Comp. Phys. Rep.*, *4*, 183.
Khazanov, G. V., M. W. Liemohn, E. N. Krivorutsky, J. M. Albert, J. U. Kozyra, and B. E. Gilchrist (1999), Relativistic electron beam propagation in the Earth's magnetosphere, *J. Geophys. Res.*, *104*, 28,587.
Khazanov, G. V., M. W. Liemohn, E. N. Krivorutsky, J. U. Kozyra, J. M. Albert, and B. E. Gilchrist (2000), On the influence of the initial pitch angle distribution on relativistic electron beam dynamics, *J. Geophys. Res.*, *105*, 16,093.
Khazanov, G. V., K. V. Gamayunov, V. K. Jordanova, and E. N. Krivorutsky (2002), A self-consistent model of the interacting ring current ions and electromagnetic ion cyclotron waves, initial results: Waves and precipitating fluxes, *J. Geophys. Res.*, *107*(A6), 1085, doi:10.1029/2001JA000180.
Khazanov, G. V., M. W. Liemohn, T. S. Newman, M.-C. Fok, and R. W. Spiro (2003a), Self-consistent magnetospheric-ionosphere coupling: Theoretical studies, *J. Geophys. Res.*, *108*(A3), 1122, doi:10.1029/2002JA009624.
Khazanov, G. V., K. V. Gamayunov, and V. K. Jordanova (2003b), Self-consistent model of magnetospheric ring current and electromag-

netic ion cyclotron waves: The 2–7 May 1998 storm, *J. Geophys. Res.*, *108*(A12), 1419, doi:10.1029/2003JA009856.
Kozyra, J. U., C. E. Rasmussen, R. H. Miller, and L. R. Lyons (1994), Interaction of ring current and radiation belt protons with ducted plasmaspheric hiss 1. Diffusion coefficients and timescales, *J. Geophys. Res.*, *99*, 4069.
Kozyra, J. U., C. E. Rasmussen, R. H. Miller, and E. Villalon (1995), Interaction of ring current and radiation belt protons with ducted plasmaspheric hiss 2. Time evolution of the distribution function, *J. Geophys. Res.*, *100*, 21,911.
Liemohn, M. W., G. V. Khazanov, and J. U. Kozyra (1997), Guided plasmaspheric hiss interactions with superthermal electrons 1. Resonance curves and timescales, *J. Geophys. Res.*, *102*, 11,619.
Lyons, L. R. (1974a), General relations for resonant particle diffusion in pitch angle and energy, *J. Plasma Phys.*, *12*, 45.
Lyons, L. R. (1974b), Pitch angle and energy diffusion coefficients from resonant interactions with ion-cyclotron and whistler waves, *J. Plasma Phys.*, 12, 417.
Lyons, L. R., and R. M. Thorne (1973), Equilibrium structure of radiation belt electrons, *J. Geophys. Res.*, *78*, 2142.
Lyons, L. R., R. M. Thorne, and C. F. Kennel (1972), Pitch-angle diffusion of radiation belt electrons within the plasmasphere, *J. Geophys. Res.*, *77*, 3455.
Meredith, N. P., R. B. Horne, and R. R. Anderson (2001), Substorm dependence of chorus amplitudes: Implications for the acceleration of electrons to relativistic energies, *J. Geophys. Res.*, *108*, 13,165.
Meredith, N. P., R. B. Horne, R. H. A. Iles, R. M. Thorne, D. Heynderickx, and R. R. Anderson (2002), Outer zone relativistic electron acceleration associated with substorm-enhanced whistler mode chorus, *J. Geophys. Res.*, *107*(A7), 1144, doi:10.1029/2001JA900146.
Meredith, N. P., M. Cain, R. B. Horne, R. M. Thorne, D. Summers, and R. R. Anderson (2003), Evidence for chorus-driven electron acceleration to relativistic energies from a survey of geomagnetically disturbed periods, *J. Geophys. Res.*, *108*(A6), 1248, doi:10.1029/2002JA009764.
Réveillé, T., P. Bertrand, A. Ghizzo, F. Simonet, and N. Baussart (2001), Dynamic evolution of relativistic electrons in the radiation belts, *J. Geophys. Res.*, *106*, 18,883.
Richtmyer, R. D., and K. W. Morton (1967), *Difference Methods for Initial-Value Problems*, John Wiley, Hoboken, N. J.
Roberts, C. S., and S. J. Buchsbaum (1964), Motion of a charged particle in a constant magnetic field and a transverse electromagnetic wave propagating along the field, *Phys. Rev.*, *135*, A381.
Schulz, M. (1991), The magnetosphere, in *Geomagnetism*, vol. 4, edited by J. A. Jacobs, pp. 87–293, Academic, San Diego, Calif.
Schulz, M. and L. J. Lanzerotti (1974), *Particle Diffusion in the Radiation Belts*, Springer-Verlag, New York.
Stix, T. H. (1962), *The Theory of Plasma Waves*, McGraw-Hill, New York.
Summers, D., and C. Ma (2000), A model for generating relativistic electrons in the Earth's magnetosphere based on gyroresonant wave-particle interactions, *J. Geophys. Res.*, *105*, 2625.
Summers, D., and R. M. Thorne (2003), Relativistic electron pitch-angle scattering by electromagnetic ion cyclotron waves during geomagnetic storms, *J. Geophys. Res.*, *108*(A4), 1143, doi:10.1029/2002JA009489.
Summers, D., R. M. Thorne, and F. Xiao (1998), Relativistic theory of wave-particle resonant diffusion with application to electron acceleration in the magnetosphere, *J. Geophys. Res.*, *103*, 20,487.
Summers, D., C. Ma, N. P. Meredith, R. B. Horne, R. M. Thorne, D. Heynderickx, and R. R. Anderson (2002), Model of the energization of outer-zone electrons by whistler-mode chorus during the October 9, 1990 geomagnetic storm, *Geophys. Res. Lett.*, *29*(24), 2174, doi:10.1029/2002GL016039.
Swanson, D. G. (1989), *Plasma Waves*, Academic, San Diego, Calif.

J. M. Albert, Air Force Research Laboratory/VSBX, 29 Randolph Road, Hanscom AFB, MA 01731-3010, USA. (jay.albert@hanscom.af.mil)